Task 1 Exact Optimisation

Jon-Paul Boyd (P17231743)
School of Computer Science and Mathematics
De Montfort University
United Kingdom

1 Exact Optimisation In One Variable

The objective of this section is to find the global minimum of 3 functions that use one variable. Note the following notation shall be used in this section.

$f\prime(x) = \frac{df(x)}{dx}$

1.1 Problem 1

Find the global minimum of the following function:

$f(x) = \sqrt{(x)},\ \forall x \in [0, 10]$

1.2 Solution 1

1.2.1 Step 1 - Find first order derivative

Finding the global minimum begins with determining the derivative (slope) of the line, with f prime of x equal to square root of x prime:

$f\prime(x) = (\sqrt{x})\prime$

The square root of x is the same as x to the half power, so now rewritten using the basic derivative power rule:

$f\prime(x) = (x^{\frac{1}{2}})\prime = \frac{1}{2} \cdot x^{\frac{1}{2}-1}$

Therefore:

$f\prime(x) = \frac{1}{2} \cdot x^{\frac{-1}{2}} = \frac{1}{2} \cdot \frac{1}{x^{\frac{1}{2}}} = \frac{1}{2} \cdot \frac{1}{\sqrt{x}}$

Consequently, the first order derivative of the given function is:

$f\prime(x) = \frac{1}{2} \cdot \frac{1}{\sqrt{x}}$

1.2.2 Step 2 - Find critical points

x values where the derivative is 0 or undefined determines the location of critical points where extrema occur. The first order derivative expression below is used:

$f\prime(x) = \frac{1}{2} \cdot \frac{1}{\sqrt{x}} = 0$

In this problem there is no value of x in the interval [0,10] that will return a first order derivative of zero. However, $x = 0$ will return a first order derivative of undefined, hence it is a critical point.

1.2.3 Step 3 - Find second order derivative

The second order derivative can be determined by taking the first order derivative of the first order derivative of the given function:

$f\prime\prime(x) = (f\prime(x))\prime = (\frac{1}{2} \cdot \frac{1}{\sqrt{x}})\prime$

As the second part of the expression is 2 terms divided, the quotient rule can be used, defined as follows:

$$(\frac{h(x)}{g(x)})\prime = \frac{h\prime(x)\cdot g(x) - h(x)\cdot g\prime(x)}{\left(g(x)\right)^2}$$

Which gives:

$$f\prime\prime(x) = \left(\frac{1}{2}\right)\cdot\frac{1\prime\cdot\sqrt{x} - 1\cdot\left(\sqrt{x}\right)\prime}{\left(\sqrt{x}\right)^2}$$

Solving the derivatives 1 by 1 using the basic derivative rules we get:

$$f\prime\prime(x) = \left(\frac{1}{2}\right)\cdot\frac{0\cdot\sqrt{x} - 1\cdot\left(\sqrt{x}\right)\prime}{\left(\sqrt{x}\right)^2} = \left(\frac{1}{2}\right)\cdot\frac{-1\cdot\left(\frac{1}{2}\cdot\frac{1}{\sqrt{x}}\right)}{x} = \left(\frac{1}{2}\right)\cdot\left(\frac{-\left(\frac{1}{2}\cdot\frac{1}{\sqrt{x}}\right)}{x}\right) = \left(\frac{1}{2}\right)\cdot\left(\frac{-\frac{1}{2\sqrt{x}}}{\frac{x}{1}}\right) = -\frac{1}{4\sqrt{x}\cdot x}$$

$$f\prime\prime(x) = -\frac{1}{4\sqrt{x}\cdot x}$$

1.2.4 Step 4 - Function, derivative and bounds analysis

The table below lists a subset of x values, including the interval bounds, with the lower bound also the identified critical point for the given function. Also listed is the calculated y value for each x for the given function. The absolute minimum of the function is located at $x = 0$. As the interval endpoints may be absolute minimums they are checked. In this case the lower bound 0 is both a critical point and the absolute minimum.

x	y
0	undefined
1	1
3	1.73
5	2.23
7	2.64
9	3
10	3.16

A first derivative sign chart (Fig. 1) can indicate where the function is increasing and decreasing. We know x=0 is both a critical point and the lower bound of the interval. The value of the first order derivative is shown above the x value number line in blue. The derivative values for all x are positive, meaning f(x) is increasing throughout the given interval.

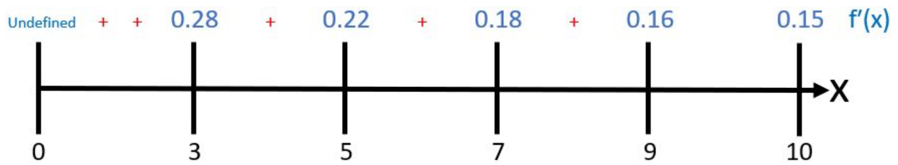

Figure 1: First derivative sign chart of $f(x) = \sqrt{(x)}$ in the interval [0, 10]

A second derivative sign chart can indicate where the function is concave up or concave down. The sign chart (Fig. 2) shows that the second derivative is always positive for the given interval, but the derivative is decreasing, meaning the function is concave down.

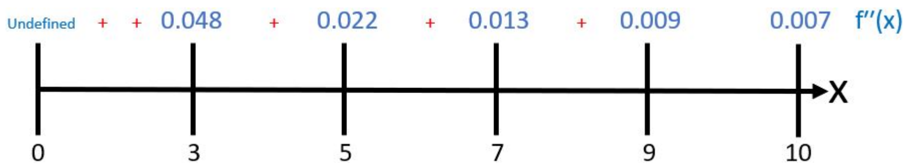

Figure 2: Second derivative sign chart of $f(x) = \sqrt{(x)}$ in the interval [0, 10]

Fig. 3 shows the plot for the given function over the interval [0, 10] to visually represent it. It confirms $f(x)$ has one global minimum when $x = 0$ (lower bound). It also confirms the function is increasing but slows as x increases, so shaped concave down.

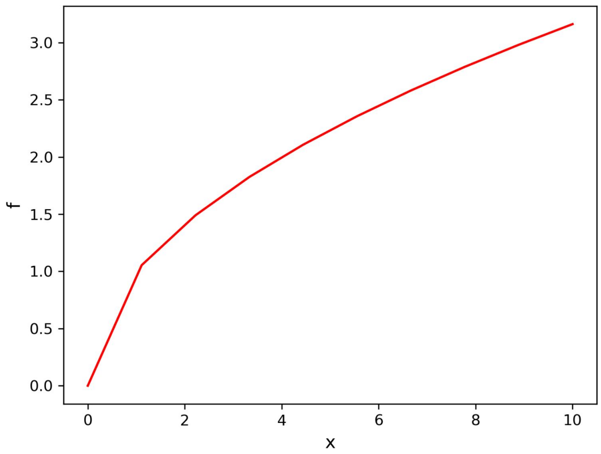

Figure 3: plot of $f(x) = \sqrt{(x)}$ in the interval [0, 10]

1.3 Problem 2

Find the global minimum of the following function:

$$f(x) = \log(x)\ \forall x \in [1, 10]$$

1.4 Solution 2

1.4.1 Step 1 - Find first order derivative

Finding the global minimum begins with determining the derivative (slope) of the line, with f prime of x equal to the natural logarithm of x prime:

$$f\prime(x) = \big(\log{(x)}\big)\prime$$

Using the natural logarithm derivative:

$$f\prime(x) = \frac{1}{x}$$

1.4.2 Step 2 - Find critical points

Following the same process as applied to problem 1, we are interested in x values where the derivative is 0 or undefined, as such values determine the location of critical points. As with problem 1, there is no value of x in the interval that will give a first derivative of zero. Furthermore, there are no values of x in the interval that return *undefined* from the first derivative. What remains to be checked are the bounds.

1.4.3 Step 3 - Find second order derivative

Take the first order derivative of the first order derivative:

$$f\prime\prime(x) = \big(f\prime(x)\big)\prime = \big(\frac{1}{x}\big)\prime$$

Using the quotient rule:

$$f\prime\prime(x) = \big(\frac{1}{x}\big)\prime = \frac{0 \cdot x - 1 \cdot 1}{x^2} = -\frac{1}{x^2}$$
$$f\prime\prime(x) = -\frac{1}{x^2}$$

1.4.4 Step 4 - Function, derivative and bounds analysis

The table below lists a subset of x values, including the interval bounds. Also listed is the calculated y value for each x for the given function. $x = 1$ is the lower bound, and is where the global minimum is located.

x	y
1	0
3	1.09
5	1.60
7	1.94
9	2.19
10	2.30

The first derivative sign chart is given in Fig. 4. x=1 is the lower bound of the interval and global minimum. The value of the first order derivative is shown above the x value number line in blue. The derivative values for all x are positive, meaning f(x) is increasing throughout the given interval.

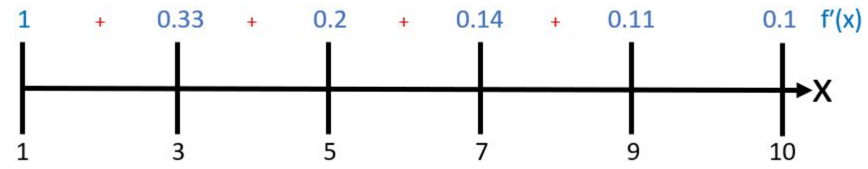

Figure 4: First derivative sign chart of $f(x) = \log(x)$ in the interval [1, 10]

A second derivative sign chart (Fig. 5) shows that the second derivative is always positive for the given interval, but the derivative is decreasing, meaning the function is concave down.

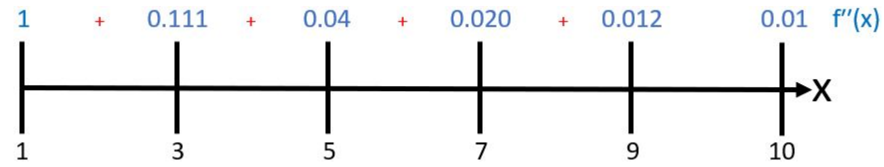

Figure 5: Second derivative sign chart of $f(x) = \log(x)$ in the interval [1, 10]

Fig. 6 shows the plot for the given function over the interval [1, 10] to visually represent it. It confirms $f(x)$ has one global minimum when $x = 1$ (lower bound). It also confirms the function is increasing but slows as x increases, so shaped concave down.

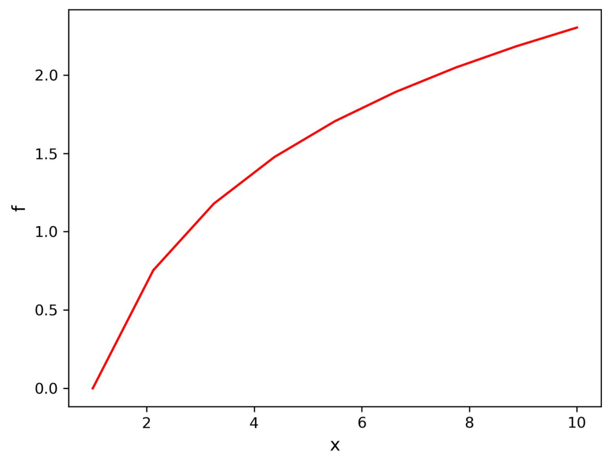

Figure 6: plot of $f(x) = \log(x)$ in the interval [1, 10]

1.5 Problem 3

Find the global minimum in the following function. Note that 2 different intervals are given. Solution steps 1-3 are the same for both intervals.

$f(x) = x^3 \; \forall \; x \in \mathbb{R}$ and $x \in [-1, 1]$

1.6 Solution 3

1.6.1 Step 1 - Find first order derivative

Finding the global minimum begins with determining the derivative (slope) of the line, with f prime of x equal to x cubed prime:

$$f\prime(x) = (x^3)\prime$$

Using the basic derivative power rule:

$$f\prime(x) = 3x^{3-1} = 3x^2$$

1.6.2 Step 2 - Find critical points

Find what x values give a derivative of 0 or undefined, as at these locations we can find extreme values.

$f\prime(x) = 3x^2 = 0$
$x = 0$ is a critical number

1.6.3 Step 3 - Find second order derivative

Take the first order derivative of the first order derivative:

$$f\prime\prime(x) = \big(f\prime(x)\big)\prime = \big(3x^2\big)\prime$$

Using the basic derivative power rule once more:

$$f\prime\prime(x) = 3 \cdot 2x^{2-1} = 6x$$

1.6.4 Step 4 - Function, derivative and bounds analysis

The interval [-1, 1] is first considered. The table below lists a subset of x values, including the interval bounds. Also listed is the calculated y value for each x from the given function. x=-1 is the lower bound, and where the lowest y value where the global minimum is located. As determined in step 2, where $x = 0$ is a critical, stationary point.

x	y
-1	-1
-0.7	-0.343
-0.4	-0.064
-0.1	-0.001
0	0
0.1	0.001
0.4	0.064
0.7	0.343
1	1

The first derivative sign chart is given in Fig. 7. x = -1 is the lower bound of the interval and global minimum. The value of the first order derivative is shown above the x value number line in blue. The derivative values for all x are positive, meaning f(x) is increasing throughout the given interval [-1, 1].

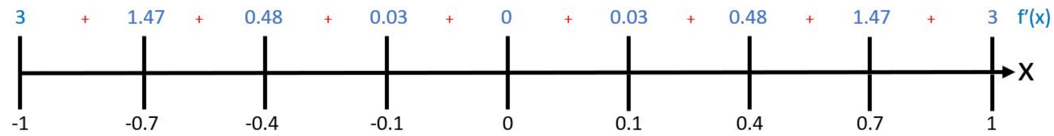

Figure 7: First derivative sign chart of $f(x) = x^3$ in the interval [-1, 1]

A second derivative sign chart (Fig. 8) shows that the second derivative is negative between [-1, 0]. This also means the first derivative is decreasing (Fig. 7) and therefore the slope is decreasing and concave down, confirmed in Fig. 9. The second derivative is positive between [0, 1] therefore the first derivative is increasing (Fig. 7) and hence the slope is increasing and concave up in this same interval (Fig. 9).

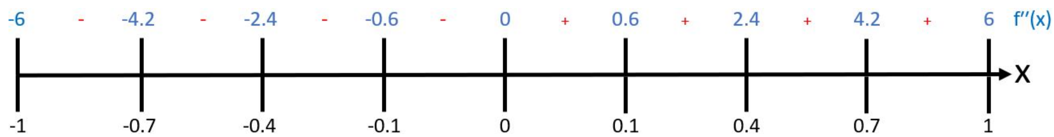

Figure 8: Second derivative sign chart of $f(x) = x^3$ in the interval [-1, 1]

Fig. 9 shows the plot for the given function over the interval [-1, 1] to visually represent it. It confirms $f(x)$ has one global minimum when x = -1 (lower bound), and has a derivative of zero at the stationary, inflection point where x = 0 and where the concavity changes.

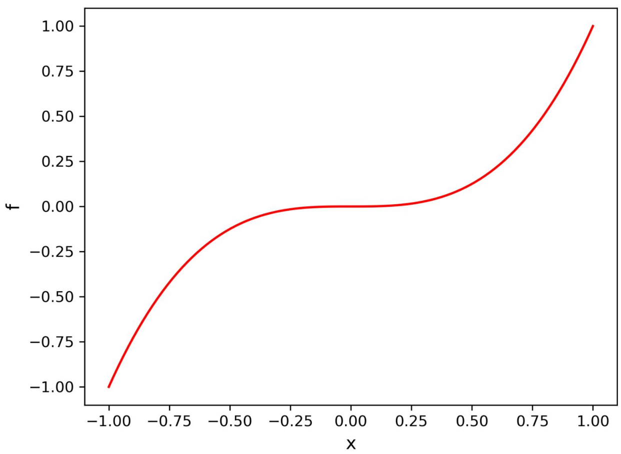

Figure 9: plot of $f(x) = x^3$ in the interval [-1, 1]

Regarding the interval $\mathbb{R}$, it is an infinite interval where the function goes up and down to $+/-\infty$. An absolute, global minimum (and maximum) does not exist. This is confirmed by evaluating the function as x approaches negative infinity:

$$\lim_{x \to -\infty} [f(x) = x^3] = (-\infty)^3 = -\infty$$

2 Exact Optimisation In Many Variables

The objective remains finding global minimum as we move up a dimension to optimise a multivariate function.

2.1 Problem 4

Find the global minimum of the following function:

$f(x, y) = 4x^3 + y^3 - 6xy$ in $\mathbb{R}^2$

Note we have 2 independent variables, x and y, defining a surface. The following notation shall be used:

$f_x\prime(x, y) = \frac{df(x,y)}{dx}$

2.2 Solution 4

2.2.1 Step 1 - Find first order partial derivatives

The first partial derivatives for x and y will help us find the peaks (maximums) and valleys (minimums) on the surface.

$f_x\prime(x, y) = (4x^3 + y^3 - 6xy)\prime = (4x^3)\prime + (y^3)\prime + (-6xy)\prime =$
$(4 \cdot 3x^{3-1}) + 0 - 6 \cdot (x\prime \cdot y + x \cdot y\prime) =$
$12 \cdot x^2 - 6 \cdot (y)$
$f_x\prime(x, y) = 12x^2 - 6y$

$f_y\prime(x, y) = (4x^3 + y^3 - 6xy)\prime = (4x^3)\prime + (y^3)\prime + (-6xy)\prime =$
$0 + 3y^{3-1} - 6 \cdot (x\prime \cdot y + x \cdot y\prime) =$
$f_y\prime(x, y) = 3y^2 - 6x$

2.2.2 Step 2 - Find critical points

Find what x and y values give partial derivatives of 0 or undefined.

$f_x\prime(x, y) = 12x^2 - 6y = 0$
$f_y\prime(x, y) = 3y^2 - 6x = 0$

The critical points are (x_1, y_1) = (0,0) and (x_2, y_2) = (0.793, 1.259)

2.2.3 Step 3 - Find second order derivative

The second derivative test will indicate if the critical points are minimums, maximums or saddle points:

$f_{xx}\prime\prime(x, y) = (12x^2 - 6y)\prime = 12 \cdot 2x = 24x$

$f_{yy}\prime\prime(x, y) = (3y^2 - 6x)\prime = 3 \cdot 2y = 6y$

$f_{xy}\prime\prime(x, y) = -6$

The hessian matrix of second order partial derivatives for the given function critical point (0.793, 1.259) is shown below:

$$\mathbf{H}_f(x, y) = \begin{bmatrix} f_{xx}\prime\prime(x, y) & f_{xy}\prime\prime(x, y) \\ f_{yx}\prime\prime(x, y) & f_{yy}\prime\prime(x, y) \end{bmatrix} = \begin{bmatrix} 24x & -6 \\ -6 & 6y \end{bmatrix}$$

$$\mathbf{H}_f(0.793, 1.259) = \begin{bmatrix} 19.032 & -6 \\ -6 & 7.554 \end{bmatrix}$$

det$\mathbf{H}_f(0.793, 1.259) = 107.767 > 0$

The second derivative test is positive and second derivative test with respect to f_{xx}'' is also positive. This means we have a minimum (concave up) that occurs at this point. We can take this location and plug it into the original function to get the height:

$$f(x,y) = 4x^3 + y^3 - 6xy =$$
$$f(0.793, 1.259) = 1.994 + 1.995 - 5.990 = -2.001$$

Therefore there is a minimum at $f(x,y) = 4x^3 + y^3 - 6xy$ = -2.001 at (x,y) ≈ (0.793,1.259).

The hessian matrix of second order partial derivatives for the given function critical point (0, 0) is shown below:

$$\mathbf{H}_f(0,0) = \begin{bmatrix} 0 & -6 \\ -6 & 0 \end{bmatrix}$$

$$\text{det}\mathbf{H}_f(0,0) = -36 < 0$$

Therefore this second derivative test determines critical point (0,0) is a saddle point.

As the function is unbounded, when x or y approach $+/-$ ∞ the given function approaches the corresponding ∞. We can conclude the earlier located minimum at (0.793,1.259) is a relative minimum.

Fig. 10 presents the function plot, where the located relative minimum (0.793,1.259) and saddle point (0, 0) can be visualised.

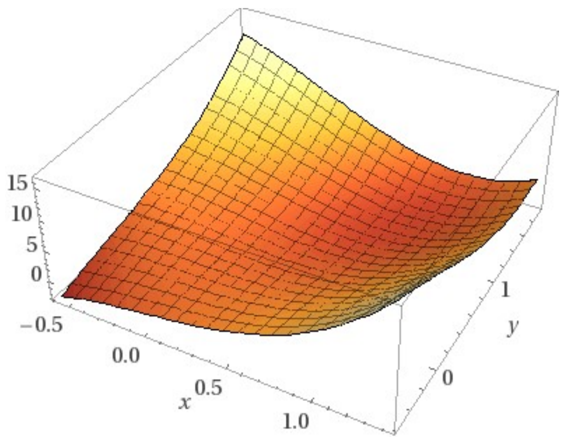

Figure 10: plot of $f(x,y) = 4x^3 + y^3 - 6xy$ in $\mathbb{R}^2$